Synchronization with positive conditional Lyapunov exponents

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Synchronization of chaotic systems may occur only when the largest conditional Lyapunov exponent of the driven system is negative. The synchronization with positive conditional Lyapunov exponents reported in a recent paper [Phys. Rev. E **56**, 2272 (1997)] is a combined result of the contracting region of the system and the finite precision in computer simulations. [S1063-651X(98)05910-8]

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Sensitivity to initial conditions is a generic feature of chaotic dynamical systems. Two chaotic orbits, starting from slightly different initial points in state space, separate exponentially with time, and become totally uncorrelated. As a result, independent identical chaotic systems cannot synchronize with each other. The sensitivity is quantitatively described by positive Lyapunov exponent(s) in the Lyapunov exponent spectrum of the chaotic system.

However, chaotic systems linked by a common signal can synchronize with each other. Several cases could be distinguished. In the first case, a replica subsystem driven by chaotic signals of the chaotic system can synchronize identically with the drive system [1–5], if the largest conditional Lyapunov is negative. This is referred to as *identical synchronization*.

Second, a driven system, which is not a replica of the drive system, may not achieve identical synchronization, but a *generalized synchronization* [6–8], if the largest conditional Lyapunov exponent is negative. Two identical systems, driven by the same signal, thus may come to the same final state due to the negative largest conditional Lyapunov exponent.

Lyapunov exponents are also employed to characterize the behavior of random dynamical systems [9]: the system is chaotic (nonchaotic) when the largest Lyapunov exponent is positive (negative). The sensitivity of a chaotic system may also be suppressed by noise, and identical chaotic systems subjected to common noise can synchronize with each other. Maritan and Banavar [10] studied the behavior of noise-driven logistic maps, and reported synchronization phenomenon. It turned out that the observed synchronization was an outcome of finite precision in numerical simulations [11,12], while the Lyapunov exponent of the noisy logistic map is positive [11].

Very recently, Shuai, Wong, and Cheng [13] claimed that synchronization can be achieved with positive conditional Lyapunov exponents. In a one-way coupled map lattice, they observed, through computer simulations, synchronization of spatiotemporal chaos with many positive components in the conditional Lyapunov exponent spectrum. Based on these results, they drew the conclusion that the conditional Lyapunov exponents cannot be used as a criterion for synchronous chaotic systems.

Whether such a claim is true is of great importance for our understanding of synchronization. In this paper, we re-examined such synchronization phenomenon, revealing that it is yet another example of round-off induced phenomenon.

In Ref. [13], Shuai, Wong, and Cheng studied a driven one-way coupled map lattice

$$y_i(t+1)=(1-\epsilon)f(y_i(t))+\epsilon f(y_{i+1}(t)) \quad (i=1,\ldots,N), \tag{1}$$

$$y_{N+1}(t)=x_0(t), \tag{2}$$

where $x_0(t)$ is a hyperchaotic signal from a one-way coupled ring lattice

$$x_0(t+1)=(1-\epsilon_0)f(x_0(t))+\epsilon_0 f(x_1(t)), \tag{3}$$

$$x_i(t+1)=(1-\epsilon)f(x_i(t))+\epsilon f(x_{i+1}(t)) \quad (i=1,\ldots,N), \tag{4}$$

$$x_{N+1}(t)=x_0(t). \tag{5}$$

The chaotic map is the well-known logistic map $f(x)=4x(1-x)$ and $\epsilon_0=0.01$.

As in Ref. [13], the conditional Lyapunov exponents of the driven system are

$$\lambda_i=\ln(1-\epsilon)+\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\ln|f'(y_i(t))|. \tag{6}$$

Let us study the simplest case of $N=1$. The conditional Lyapunov exponent as a function of ϵ is shown in Fig. 1. To detect the behavior of synchronization, 100 performances

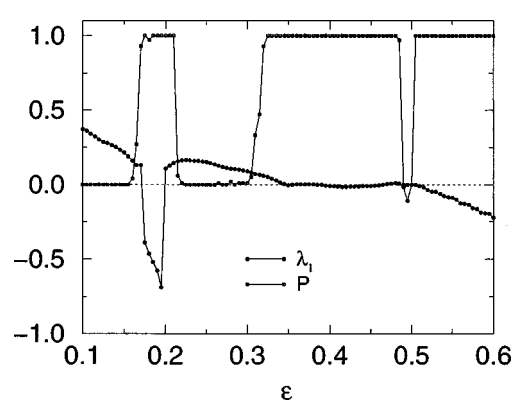

FIG. 1. Conditional Lyapunov exponent λ_1 and synchronization ratio P as functions of ϵ.

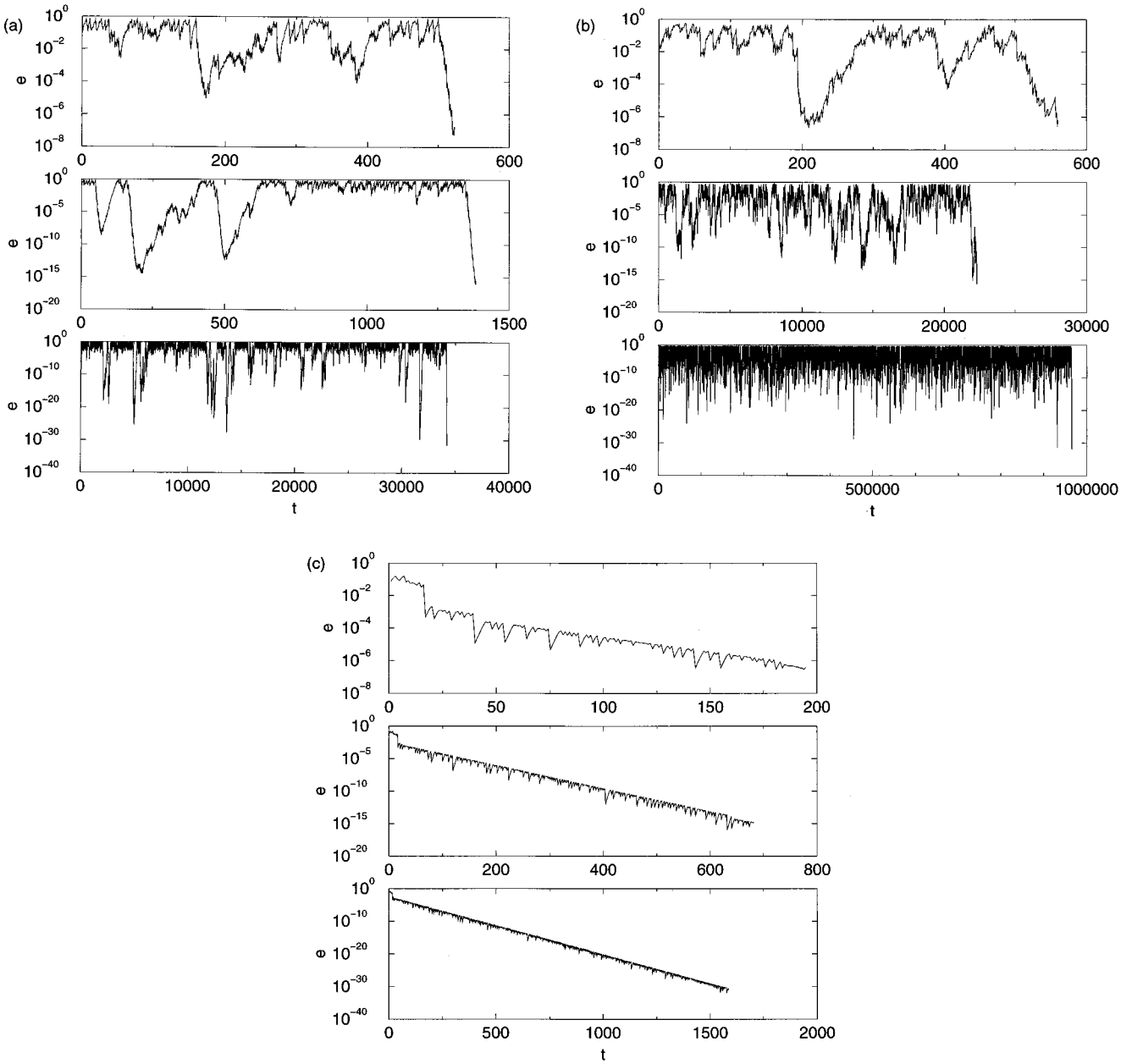

FIG. 2. The time series of the difference $e(t)$ proceeding synchronization in simulations with single, double, and quadrupole precisions. (a) $\epsilon=0.200$, SP. (b) $\epsilon=0.335$, SP. (c) $\epsilon=0.52$, SN.

with random initial conditions are carried out for each ϵ. Synchronization occurs when y_1 and x_1 become numerically identical for the finite precision in simulations (double precision). $P=M/100$, where M is the number of simulations in which synchronization occurs within 5×10^7 iterations, is estimated as a function of ϵ, as shown in Fig. 1. It can be detected that synchronization with a positive Lyapunov exponent (SP) occurs in several regions. We will take $\epsilon=0.200$ and 0.335 as examples, as pointed out in Ref. [13].

Is SP a true physical phenomenon or an artifact of finite precision in computer simulations? In the following, different precision formats (single, double, and quadruple precisions) are employed in the simulations. First, the difference $e(t)=|y_1(t)-x_1(t)|$ preceding the synchronized state is examined for simulations with different precisions but the same random initial conditions. The results for $\epsilon=0.200(\lambda_1=0.105)$ and $0.335(\lambda_1=0.025)$ are shown in Figs. 2(a) and 2(b), respectively. For comparison, an example of synchronization with negative conditional Lyapunov exponent (SN) at $\epsilon=0.520(\lambda_1=-0.041)$ is illustrated in Fig. 2(c). Note the different scales for the different precisions used. $e(t)$ displays an intermittent behavior before reaching SP. SP occurs somewhat abruptly, when $e(t)$ drops lower than the precision of the computer. The time T needed for SP to occur is much longer for quadrupole precision than that for single and double precisions. As for SN, $e(t)$ continues its trend of decrease when higher precision is employed in the simulation. It is plausible to imagine that, for SP, the intermittence of $e(t)$ will continue indefinitely for infinite precision, while, for SN, $e(t)$ will approach 0. So the physical process of SP is an intermittence, with $e(t)$ becoming very small and enlarging to the size of the chaotic attractor alternately.

The difference $e(t)$ is actually not zero even beyond the precision of the computer. In the following simulation with

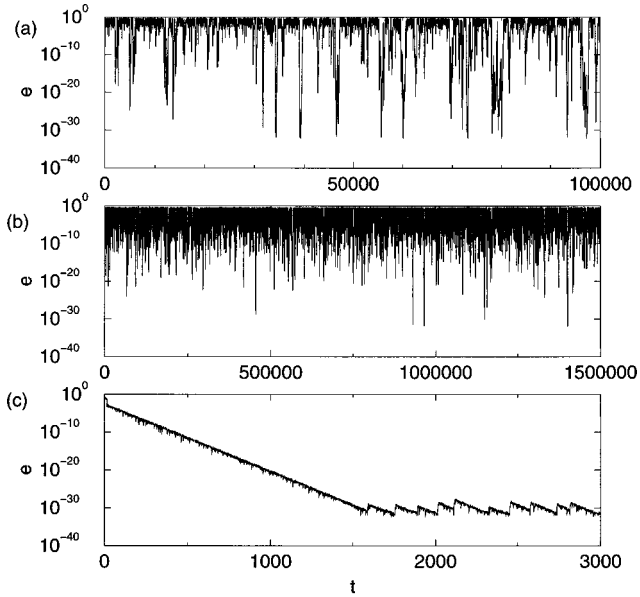

FIG. 3. The time series of the difference $e(t)$ under impulsive perturbations between -10^{-30} and 10^{-30} in simulations with quadrupole precision. The initial conditions are the same as Fig. 2. (a) $\epsilon=0.200$, SP. (b) $\epsilon=0.335$, SP. (c) $\epsilon=0.52$, SN.

quadrupole precision, when the states of the systems are numerically identical (SP), a perturbation $\xi\in(-10^{-30},10^{-30})$ is added to the drive signal $x_0(t)$ of system y at the next iteration, under the constrain $0<x_0(t)+\xi<1$. Such a tiny perturbation can totally destroy the synchronization behavior when $\lambda_1>0$, as seen from Figs. 3(a) and 3(b) for the results of $\epsilon=0.200$ and 0.335, respectively, because the tiny difference can be amplified to the order of 10^0 due to the positive conditional Lyapunov. While for SN, $e(t)$ continues to decrease after the impulsive perturbations, and the level of difference is the order of 10^{-30}. Such a dramatic difference between the behavior of SP and SN shows that a negative conditional Lyapunov is a necessary condition for physical synchronization.

To demonstrate further that synchronization cannot be achieved physically with a positive conditional Lyapunov exponent, the average synchronization time T_a is evaluated for different precisions with 100 random initial conditions. The results for SP ($\epsilon=0.200$ and 0.335) are displaced with a linear-log plot in Fig. 4(a). The three points lies almost on a straight line, meaning that $T_a\sim\exp(AL)$, where L is the number of significant digits of the finite precision. An exponential increase of T_a with L proves that synchronization can never occur with infinite precision. The behavior of SN is greatly different, where T_a follows a linear dependence on L, $T_a\sim BL$, as seen from the result of $\epsilon=0.52$ displaced with a linear-linear plot in Fig. 4(b). The reason is that, approximately, $e(t)$ decreases exponentially with time, so that $10^{-L}\sim\exp(\lambda_1 T_a)$, resulting in $B=-\ln 10/\lambda_1$. $B=56.2$ at $\epsilon=0.520$ is in good agreement with the slope 58.5 of the solid line in Fig. 4(b).

Why then can SP be observed in numerical simulations even within thousands of iterations? The origin is that there are contracting regions in a map f, $C=\{x,y||f(x)-f(y)|<1\}$. Two orbits in a contracting region come closer to each other at the next step. For the system studied above, the contracting region is $1-[1/4(1-\epsilon)]<x_1+y_1<1+[1/4(1-\epsilon)]$. The strip near $x_1+y_1=1$ has the strongest contracting rate. The distribution of x_1+y_1 is calculated with 10^7 iterations to examine the relationship between SP and C. As seen from the results of $\epsilon=0.200$ and 0.335 in Fig. 5(a), the distribution for $\epsilon=0.200$ has very high peaks in the contracting region, and a lower peak for $\epsilon=0.335$. Such a greater frequency of access to the contracting region at $\epsilon=0.200$ makes $e(t)$ drops to a much smaller value more frequently than at $\epsilon=0.335$ (see Fig. 3), which accounts for the result that SP is observed at $\epsilon=0.200$ with much fewer iterations than that at $\epsilon=0.335$. However, the evolution of the difference $e(t)$ is a combined result of local stability and instability. The finite-time Lyapunov exponent [14]

$$\lambda^{(m)}=\frac{1}{m}\sum_{t=1}^{m}\ln|f'(y_1(t))| \tag{7}$$

measures the average expansion or contraction rate in m steps. The distributions of $\lambda^{(m)}(m=70)$ for $\epsilon=0.200$ and 0.335 are illustrated in Fig. 5(b). A pronounced tail to -1.0 at $\epsilon=0.200$ means that the difference shrinks by a factor of $e^{-70}\cong 4\times 10^{-31}$ in some 70 successive iterations. The negative tails thus plays an important role in the observation of SP. It also explains the fact that SP more easily occurs at $\epsilon=0.200$ with a larger positive conditional Lyapunov exponent ($\lambda=0.105$) than at $\epsilon=0.335$ ($\lambda=0.025$). The distributions of $\lambda^{(m)}$ also reflects the true dynamics of SP: the difference $e(t)$ can be very small in a period of time, and it will be amplified in some other period of time because temporal separation dominates, thus resulting in an intermittent dy-

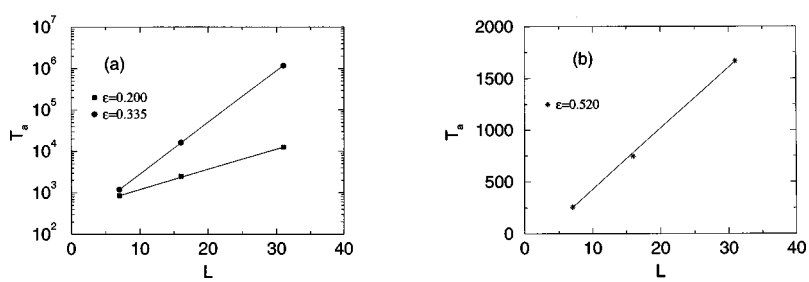

FIG. 4. Average synchronization time T_a as a function of precision in simulations. $L=7$, 16, and 31 for single, double, and quadrupole precisions, respectively. Each solid line links the first and last point of each data set. (a) Linear-log plots for SP. (b) A linear, linear plot for SN.

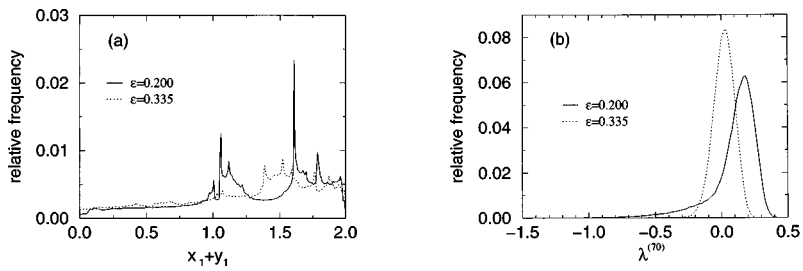

FIG. 5. (a) Normalized histograms of x_1+y_1. (b) Normalized histograms of a finite-time Lyapunov exponent.

namics. So the finite-time Lyapunov exponent gives a more convincing account for the occurrence of SP in simulations.

Based on the above analysis of the simplest case of $N=1$, the SP observed in computer simulations is a combination of two factors: the shift of the state of the chaotic system to the contracting region, and the finite precision in numerical simulations. For the case of $N>1$, SP is observed for similar reasons. The driven system is coupled in a cascade way $x_N \rightarrow x_{N-1} \rightarrow \cdots \rightarrow x$, and synchronization can only occur for the first several N_1 nodes if all the N_1 conditional Lyapunov exponents $\lambda_N, \lambda_{N-1}, \cdots \lambda_{N-N_1+1}$ are negative. Physical synchronization of all the lattices can only occur when all the conditional Lyapunov exponents are negative.

In conclusion, synchronization with positive conditional Lyapunov exponents in computer simulations is a roundoff induced phenomenon. The physical dynamics of SP is an intermittence. One can expect to observe SP easily in computer simulations in such systems with large contracting regions, and the couplings have the effect of shifting the state to the contracting regions so that the finite-time Lyapunov has a significant tail of negative values. A negative conditional Lyapunov exponent is a necessary condition for synchronizing chaotic systems.

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